cardinal number of a set $\mathcal{X}$ [9, p. 552]). This can be shown by using the fact that $b_k(t_n)$ and $b_{k'}(t_n)$ are independent of all other elements of $\mathbf{B}$ due to Assumption 1. Hence, for every possible combination of $b_k(t_n)$ and $b_{k'}(t_n)$ it is possible to group terms corresponding to $b_{k,l}(t_n) = b_k(t_n)$ and $b_{k',l}(t_n) = b_{k'}(t_n)$ on the left-hand side of (39) into a sub sum which is equal to the marginal distribution of that combination. Finally, note that since $b_k(t_n), b_{k'}(t_n) \in \{0,1\}$ and

$$\Pr\{b_k(t_n)=1, b_{k'}(t_n)=1\} = \begin{cases} (1-p_k)(1-p_{k'}), & \text{if } k \neq k' \\ (1-p_k), & \text{if } k = k' \end{cases}$$

the proof follows.

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Corrections to "Performance Analysis of the FastICA Algorithm and Cramér–Rao Bounds for Linear Independent Component Analysis"

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***Abstract*—The derivation of the Cramér–Rao bound (CRB) in ["Performance Analysis of the FastICA Algorithm and Cramér–Rao Bounds for Linear Independent Component Analysis," *IEEE Trans. Signal Process.*, vol. 54, no. 4, Apr. 2006, pp. 1189–1203] contains errors, which influence the matrix form of the CRB but not the CRB on variance of relevant off-diagonal elements of the demixing matrix. In this correspondence, we correct these errors.**

I. THE FISHER INFORMATION MATRIX FOR ICA

The referenced paper considers a standard linear independent component analysis (ICA) model of a given $d \times N$ data matrix

$$\mathbf{X} = \mathbf{AS} \tag{1}$$

where $\mathbf{A}$ is an unknown, nonsingular $d \times d$ mixing matrix. The joint probability density function (pdf) of the independent components is assumed to be $f_{\mathbf{s}}(\mathbf{S}) = \prod_{i=1}^{d}\prod_{j=1}^{N} f_i(s_{ij})$, where s_{ij} is the (i,j)th element of $\mathbf{S}$, $i = 1,\ldots,d$, $j = 1,\ldots,N$, and f_i is the pdf of s_{ij}.

The data matrix $\mathbf{X}$ is obtained as a linear transformation of $\mathbf{S}$, $\mathbf{X} = \mathbf{W}^{-1}\mathbf{S}$, or equivalently, $\text{vec}[\mathbf{X}] = (\mathbf{I}_N \otimes \mathbf{W}^{-1})\text{vec}[\mathbf{S}]$, where $\mathbf{W} = \mathbf{A}^{-1}$, $\mathbf{I}_N$ denotes the $N \times N$ identity matrix and $\otimes$ is the Kronecker product. Therefore, the joint pdf of the data has the form

$$f_{\mathbf{x}\,|\,\boldsymbol{\theta}}(X\,|\,\boldsymbol{\theta}) = |\det \mathbf{W}|^N f_{\mathbf{s}}(\mathbf{WX}) \tag{2}$$

where $\boldsymbol{\theta}$ is the unknown to-be-estimated vector parameter, $\boldsymbol{\theta} = \text{vec}[\mathbf{W}]$. The error in [1] begins with the missing exponent N in the pdf expression above; cf. [1, eq. (32)].

A straightforward computation similar to that in Appendix C in [1] follows that (34) in [1] should be replaced with

$$\mathbf{F}_{mn} = N\left(a_{ju}a_{vi} + \delta_{iu}a_{ji}a_{vi}(\eta_i - 2) + \delta_{iu}\kappa_i \sum_{\ell=1,\ell\neq u}^{d} a_{j\ell}a_{v\ell}\right). \tag{3}$$

Recall for completeness that $\mathbf{F}_{mn}$ is the mnth element of the $d^2 \times d^2$ Fisher information matrix $\mathbf{F}_\theta$, where $m = (i-1)d + j, n = (u-1)d+v$, a_{ij} denotes the ijth element of the matrix $\mathbf{A}$, $\kappa_i \stackrel{\text{def}}{=} \mathrm{E}[\psi_i^2(s_{ij})]$, $\eta_i \stackrel{\text{def}}{=} [s_i^2\psi_i^2(s_{ij})]$, and $\psi_i \stackrel{\text{def}}{=} = -f_i'/f_i$. A comparison of (3) with (34) in [1] shows that the correct Fisher information matrix element does not include any term proportional to $(N-1)^2$ but is proportional to N.

The derivation of (3) via Appendix C in [1] can be simplified by putting $N = 1$ and multiplying the resultant information matrix by N afterwards. The information matrix must be proportional to N, because

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the observed data are composed of N independent observations of a random vector with the same distribution.

The computation proceeds by proving the formula

$$\mathbf{F}_\theta = (\mathbf{A}^T \otimes \mathbf{I})\mathbf{F}_\mathbf{I}(\mathbf{A} \otimes \mathbf{I}) \tag{4}$$

where $\mathbf{F}_\mathbf{I}$ stands for the Fisher information matrix derived for a case when $\mathbf{A} = \mathbf{I}$ (identity matrix). Note the error in the matrix transpositions in (35) in [1] and also in (37). The latter equation should read $\mathbf{F}_\mathbf{G} = (\mathbf{W}^T \otimes \mathbf{I})\mathbf{F}_\theta(\mathbf{W} \otimes \mathbf{I}) = \mathbf{F}_\mathbf{I}$. A similar typo exists in [2].

For the proof of (4), it was referred to (28) in [1], but it is not accurate. In fact, (4) was only inspired by this formula.

Substituting $a_{ij} = \delta_{ij}$ into (3), it easily follows that the mnth element of $\mathbf{F}_\mathbf{I}$ for $m = (i-1)d + j$ and $n = (u-1)d + v$ reads

$$(\mathbf{F}_\mathbf{I})_{mn} = N(\delta_{ju}\delta_{vi} + \delta_{ji}\delta_{vu}\delta_{vi}(\eta_i - \kappa_i - 2) + \delta_{iu}\delta_{vj}\kappa_i). \tag{5}$$

Again, there is no term proportional to $(N-1)^2$, unlike (36) in [1]. Thus, $\mathbf{F}_\mathbf{I}$ can be written as $\mathbf{F}_\mathbf{I} = N(\mathbf{P} + \mathbf{\Sigma})$, where $\mathbf{P}$ is a permutation matrix and $\mathbf{\Sigma}$ is a diagonal matrix such that the mnth element of $\mathbf{P}$ and $\mathbf{\Sigma}$ are $\delta_{ju}\delta_{vi}$, and $\delta_{iu}\delta_{vj}[\kappa_i + \delta_{ij}(\eta_i - \kappa_i - 2)]$, respectively, for $m = (i-1)d + j$ and $n = (u-1)d + v$.

To prove (4) rigorously, note that the mnth elements of $\mathbf{A} \otimes \mathbf{I}$ and $\mathbf{A}^T \otimes \mathbf{I}$ for $m = (i-1)d + j$ and $n = (u-1)d + v$ are equal to $(\mathbf{A} \otimes \mathbf{I})_{mn} = a_{iu}\delta_{jv}$ and $(\mathbf{A}^T \otimes \mathbf{I})_{mn} = a_{ui}\delta_{jv}$, respectively. Then, the mnth element of the product $(\mathbf{A}^T \otimes \mathbf{I})\mathbf{F}_\mathbf{I}(\mathbf{A} \otimes \mathbf{I})$ is

$$[(\mathbf{A}^T \otimes \mathbf{I})\mathbf{F}_\mathbf{I}(\mathbf{A} \otimes \mathbf{I})]_{mn} = \sum_{n',n''} (\mathbf{A}^T \otimes \mathbf{I})_{mn'}(\mathbf{F}_\mathbf{I})_{n'n''}(\mathbf{A} \otimes \mathbf{I})_{n''n}. \tag{6}$$

A straightforward computation gives that the matrix element in (6) is identical to that in (3).

Appendix D in [1] should be changed accordingly. Application of the matrix inversion lemma is not needed, and the rest of the Appendix and the final result in (38) are correct. Note that an alternative elegant method of inversion of a matrix similar to $\mathbf{F}_\mathbf{I}$ was used in [2].

Finally, note that one of assumptions of the Cramér–Rao inequality is that the support of the pdf $f_{\mathbf{x}|\theta}(\mathbf{X}|\theta)$ is independent of the estimated parameter $\boldsymbol{\theta}$. In the ICA scenario, this assumption is equivalent to the condition that $f_i(x) > 0$ for all i and finite x. In particular, the CRB is not defined for sources with a bounded (limited) support.

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Hilbert Pair of Orthogonal Wavelet Bases: Revisiting the Condition

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***Abstract*—It is now well known that in order to have wavelet bases that form a Hilbert Transform pair, the corresponding low-pass conjugate quadrature filters (CQF) should ideally be related through a half sampled delay, i.e., $e^{-j\omega/2}$. In this correspondence we revisit this condition and examine some subtleties associated with this condition that were overlooked in previous work. We will show that there is a more general condition where the delay can be any "*even+half*" samples, i.e., $e^{-j(2N+1/2)\omega}$. More importantly we examine the implications in formulating design strategies for Hilbert pairs and its implementation.**

***Index Terms*—Complex wavelet, dual-tree, Hilbert pair, orthonormal filter banks..**

I. Introduction and Preliminaries

Overcomplete complex (valued) transforms that are based on the Hilbert pairs are becoming an increasingly important signal processing tool [1]. These complex transforms have the advantage of approximate shift-invariance over the critically sampled real (valued) wavelet transforms.

Orthogonal wavelets are usually associated or obtained from a low-pass conjugate quadrature filter (CQF) $H(z)$. A CQF satisfies $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1$. In the filter bank, the constituent filters, denoted by $H_0(z)$ (low-pass analysis), $H_1(z)$ (high-pass analysis), $F_0(z)$ (low-pass synthesis), and $F_1(z)$ (high-pass synthesis), are *usually* obtained from a CQF filter $H(z)$ as follows:

$$\begin{aligned} H_0(z) &= H(z) \quad H_1(z) = z^{-1}H(-z^{-1}) \\ F_0(z) &= H(z^{-1}) \quad F_1(z) = z\,H(-z). \end{aligned} \tag{1}$$

With (1), it can be verified that the aliasing function $A(z) \equiv H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0$ and the reconstruction function $T(z) \equiv H_0(z)F_0(z) + H_1(z)F_1(z) = 1$, perfect reconstruction with zero delay.

In a Hilbert pair, the two wavelets corresponding to two CQFs are related through the Hilbert transform:

$$\Psi^g(\omega) = \begin{cases} -j\Psi^h(\omega) & \text{for } \omega > 0 \\ j\Psi^h(\omega) & \text{for } \omega < 0 \end{cases} \tag{2}$$

where $\Psi^h(\omega)$ and $\Psi^g(\omega)$ are the Fourier transforms of $\psi^h(t)$ and $\psi^g(t)$, respectively. By denoting the corresponding CQFs by $H^h(z)$ and $H^g(z)$ respectively, it was first shown in [2] that (2) is achieved if

$$H^g(e^{j\omega}) = e^{-j\omega/2}H^h(e^{j\omega}), \quad |\omega| < \pi \tag{3}$$

and is known as the *half sample delay condition*. Further analysis on the condition were presented in [3] and [4] using alternative formulations which are easier to manipulate analytically. The conclusion drawn in [3] and [4] are similar to that in [2], namely the half sample delay condition is required. The most general analysis appeared in [4] where no assumption on the relationship between the two CQFs were made and (3) is shown to be necessary and sufficient.

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